On Rigidly Scalar-Flat Manifolds

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Abstract

Witten and Yau (hep-th/9910245) have recently considered a generalisation of the AdS/CFT correspondence, and have shown that the relevant manifolds have certain physically desirable properties when the scalar curvature of the boundary is positive. It is natural to ask whether similar results hold when the scalar curvature is zero. With this motivation, we study compact scalar flat manifolds which do not accept a positive scalar curvature metric. We call these manifolds rigidly scalar-flat. We study this class of manifolds in terms of special holonomy groups. In particular, we prove that if, in addition, a rigidly scalar flat manifold M is *Spin* with $\dim M \geq 5$, then M either has a finite cyclic fundamental group, or it must be a counter example to Gromov-Lawson-Rosenberg conjecture.

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1 Introduction

1/1. Motivation: Physics. Witten and Yau [28] have recently considered the generalization of the celebrated AdS/CFT correspondence to the case of a complete $(n+1)$-dimensional Einstein manifold W of negative Ricci curvature with a compact conformal (in the sense of Penrose) boundary M of dimension n. (In the simplest case, W is a hyperbolic space, and M is the standard sphere.) It can be shown that the relevant conformal field theory on M is stable if the scalar curvature R_M of M is positive, but not if R_M can be negative. When $R_M = 0$, stability is more delicate; the theory is stable in some cases, but not in the others. Witten and Yau [28] show that if R_M is positive, then W and M have several physically desirable properties: the boundary M is connected, and W is free of wormholes. Furthermore, the topology of M exerts a strong influence on that of W (the fundamental group of W is no “larger” than that of M).

We wish to ask whether similar statements are valid when $R_M = 0$ but the conformal field theory on M is still stable. First, suppose that M is such that the conformal structure can be perturbed so that the scalar curvature becomes positive everywhere; it is obvious that the Witten-Yau conclusions then hold. *Therefore, the only non-trivial case is the one in which such perturbations are not possible.* We shall say in the latter case that M is *rigidly* scalar-flat. To summarize, then, the *rigidly* scalar-flat boundaries are the only cases in which the difficulties discussed by Witten and Yau could possibly arise.

For the benefit of physically oriented readers, we can expand on this crucial point as follows. Let M be any scalar-flat compact manifold. Let x be a point in M, and perturb the metric slightly (but *not* conformally) so that the scalar curvature at x is an arbitrarily small positive number. This perturbation can give rise to two possibilities: either it maintains $R_M \geq 0$ everywhere on M, or it does not. (That is, in the second case, M is such that the scalar curvature *must* become negative at some other point if it becomes positive at x.) In the latter case we say that M is *rigidly scalar flat*, while in the former it is non-rigid. Now suppose that M is *not* rigidly scalar-flat, and that we slightly perturb as above so that $R_M \geq 0$ everywhere, and $R_M(x) > 0$. Then according to a theorem of Kazdan and Warner [9], by means of a conformal deformation of the metric we can force the scalar curvature to be positive *everywhere*. (Recall that the metric on M is in any case only defined modulo conformal factors, so this procedure changes nothing physical.) But now we apply the result of Witten and Yau [28]. *We conclude that the results of Witten and Yau are valid even when $R_M = 0$, provided that M is* not *rigidly scalar-flat.*

At this point it is clear that we need two things: first, a way of distinguishing compact rigidly scalar-flat manifolds from those which are not rigid; and second, a way of extending the results of Witten and Yau to the rigid case.

The objective of this work is to supply the first of these. Our results essentially state that, given the truth of the Gromov-Lawson-Rosenberg conjecture, compact, locally irreducible, rigidly scalar flat *Spin* manifolds have structures which are very special and can be described very precisely. We hope that this precise description of these manifolds will facilitate a completion of the Witten-Yau results.

The reader will notice in particular that the fundamental group of M is severely constrained: it must be a cyclic group of low order. Presumably this has important consequences for W. On a less conjectural note, manifolds *not* in the lists given in Theorem 2.10 below are (given certain easily verified conditions) *not* rigidly scalar flat. Thus, for example, the familiar six-dimensional Calabi-Yau quintic in $\mathbf{CP}^4$ is obviously scalar-flat, but it is not rigidly scalar flat: so the Witten-Yau conclusions are valid in this case.

1/2. Motivation: Geometry. Henceforth we consider only smooth, closed and compact manifolds. As above, a scalar-flat Riemannian manifold is called *rigidly scalar-flat* if it accepts no metric of positive scalar curvature. The manifold is "rigid" in the sense that if a scalar-flat metric is perturbed so that the scalar curvature becomes positive at some point, then it must become negative at some other point. A basic theorem of Bourguignon [2, Corollary 4.49] states that for a rigidly scalar-flat manifold the space of Ricci-flat metrics coincides with the space of scalar-flat metrics.

We recall a well-known fact: that all closed manifolds (of dimension at least three) are divided into the following three classes (see [9]):

- **(P)** manifolds accepting a positive scalar curvature metric;
- **(Z)** manifolds accepting a non-negative scalar curvature metric, but not accepting a positive scalar curvature metric;
- **(N)** manifolds not accepting a metric of a non-negative scalar curvature.

It follows from [9] that the class of rigidly scalar-flat manifolds is nothing but the class **(Z)**. Our objective in this paper is to study and emphasize general properties of rigidly scalar-flat manifolds.

Since these manifolds are necessarily Ricci-flat, it follows from the Cheeger-Gromoll theorem that they have the structure $(T^r \times \widehat{M})/F$, where T^r is a flat torus, and F is a finite group and so is the fundamental group of $\widehat{M}$. By the

work of Gromov-Lawson [6], $T^r \times \widehat{M}$ is rigidly scalar-flat if and only if $\widehat{M}$ is rigidly scalar-flat, so we see that the study of the structure of compact rigidly scalar-flat manifolds reduces to understanding the behavior of the rigidity condition under *finite* coverings and quotients. It is therefore natural to begin by removing the toral factor.

We shall consider manifolds which are compact and *locally irreducible*. That is, the universal Riemannian cover is not isometric to a product of lower-dimensional manifolds. If $\dim M > 1$, this eliminates the toral factor, and forces the fundamental group $\pi_1(M)$ to be finite. In addition, we assume that all manifolds we consider are oriented.

When the manifold is simply connected, and $\dim M \geq 5$, there are two very different cases:

- M is not a *Spin* manifold,
- M is a *Spin* manifold.

In the first case the work of Gromov-Lawson [5] implies that there are no rigidly scalar-flat manifolds, since a simply connected non-*Spin* manifold (with $\dim M \geq 5$) always accepts a metric of positive scalar curvature. In the *Spin* case Stolz' theorem [24] says that M (again, if $\dim M \geq 5$) does not accept a positive scalar curvature metric if and only if $\alpha(M) \neq 0$ in the real K-theory KO_n. Here, recall that α is the index of the Dirac operator valued in the real K-theory. In dimensions equal to a multiple of 4 we use the notation $\hat{A}$ instead of α.

According to Besse, an n-dimensional locally irreducible Riemannian manifold has *generic* holonomy if its restricted holonomy group is isomorphic to $SO(n)$, or to $U(\frac{n}{2})$, if it is locally Kählerian. Otherwise it has *special* holonomy. It was observed by Futaki [4] that Stolz' theorem implies that a rigidly scalar-flat compact *Spin* simply connected manifold has special holonomy. But Besse emphasizes that such manifolds are Einstein, and, of course, a scalar-flat Einstein manifold is Ricci-flat. Thus we see that Bourguignon's theorem should be understood in terms of the concept of special holonomy. We believe that this phenomenon underlies Bourguignon's theorem in general: that is, locally irreducible rigidly scalar-flat compact manifolds are Ricci-flat **because** they have special holonomy. That is, we conjecture

Conjecture 1.1 (Weak Besse Conjecture) Every locally irreducible rigidly scalar-flat compact manifold has special holonomy.

Remark 1.2 Besse mentions [2, p. 19] the possibility that *all* locally irreducible compact Ricci-flat manifolds have special holonomy. We call this statement Besse's Conjecture.

2 Some basic results

We emphasize that we consider here *Spin* manifolds. Now as a guide to deciding what one can reasonably expect to prove, we have

Lemma 2.1 *Let M be a compact locally irreducible rigidly scalar-flat Spin manifold, of dimension ≥ 5. Suppose either that*

(a) $\dim M$ *is not a multiple of* 4,

or **(b)** $\dim M$ *is a multiple of* 4, *but* $\pi_1(M)$ *is not cyclic.*

Then M is a counter-example to the Gromov-Lawson-Rosenberg Conjecture.

Before proving Lemma 2.1, we briefly recall a few things. Let $\pi_1 M = \pi$ be a finite group; then a classifying map $f : M \longrightarrow B\pi$ defines the element $[(M, f)] \in \Omega_n^{Spin}(B\pi)$, where $\Omega_n^{Spin}(\cdot)$ is the *Spin* cobordism theory. Then α is the composition

$$\alpha : \Omega_n^{Spin}(B\pi) \xrightarrow{D} KO_n(B\pi) \xrightarrow{A} KO_n(C_r^*\pi),$$

where D is the Atiyah-Bott-Shapiro map, and A is the assembly map; see [19, 20, 21] for details.

The K-theory groups $KO_*(C_r^*\pi)$ and the assembly map A are well-understood when π is a finite group; see [21]. In particular, the group $KO_n(C_r^*\pi)$ and the invariant $\alpha(M)$ may be described as follows. Since $\pi = \pi_1 M$ is a finite group, we consider the universal cover $\widetilde{M} \longrightarrow M$. A *Spin* structure on M (let $\dim M = n$) lifts to $\widetilde{M}$, and gives the Dirac operator D on $\widetilde{M}$, which commutes with the action of the Clifford algebra $\mathcal{C}\ell_n$ and the deck-transformations of π. This turns the kernel $\mathrm{Ker}(D)$ of the Dirac operator D into a module over the algebra $\mathcal{C}\ell_n \otimes \mathbf{R}\pi$, where $\mathbf{R}\pi$ is the real group ring of π. Let $i : \mathcal{C}\ell_n \to \mathcal{C}\ell_{n+1}$ be the natural inclusion, and let $\mathbf{G}(\mathcal{C}\ell_j \otimes \mathbf{R}\pi)$ be the corresponding Grothendieck group. Then the invariant $\alpha(M)$ is nothing but the residue class of $\mathrm{Ker}(D)$ in the corresponding K-theory:

$$\alpha(M) = [\mathrm{Ker}(D)] \in \mathbf{G}(\mathcal{C}\ell_n \otimes \mathbf{R}\pi)/i^* \mathbf{G}(\mathcal{C}\ell_{n+1} \otimes \mathbf{R}\pi) = KO_n(\mathbf{R}\pi) = KO_n(C_r^*\pi).$$

Thus the K-theory $KO_n(\mathbf{R}\pi)$ is "assembled" out of regular KO_*-groups, parametrized by representations of π. In particular, if $\alpha(M) \neq 0$ it means that

there exists a harmonic spinor on $\widetilde{M}$. Taking into account that the Gromov-Lawson-Rosenberg Conjecture fails for non-finite fundamental groups, we state it in the following form:

Conjecture 2.2 (Gromov-Lawson-Rosenberg Conjecture) A compact *Spin* manifold M with $\dim M \geq 5$ and finite fundamental group does not accept a positive scalar curvature metric if and only if $\alpha(M) \neq 0$.

Remark 2.3 Under the conditions of Lemma 2.1 a manifold M is compact, locally irreducible, and Ricci-flat,so $\pi_1(M)$ is finite. Thus the Gromov-Lawson-Rosenberg Conjecture predicts under these circumstances that rigidity implies $\alpha(M) \neq 0$.

Proof of Lemma 2.1. Case (a) Assume M is not a counter-example. Then $\alpha(M) \neq 0$, and M admits a parallel spinor, and so it follows from Futaki's paper [4, Proposition 2.3] that the holonomy must be special. Now a compact *Spin* manifold of dimension $n \neq$ multiple of 4 always has $\alpha(M) = 0$ except in dimensions $8k+1$ and $8k+2$. There are no scalar-flat locally irreducible manifolds of special holonomy in odd dimensions other than 7 (by the Berger-Simon theorem (see [2, Chapter 10]) and the fact that scalar-flat symmetric spaces are flat). Finally, it follows from [16] that a (compact, orientable) scalar-flat locally irreducible manifold of special holonomy in dimension $8k+2$ is necessarily of holonomy $SU(4k+1)$.

Note that this is the *full* holonomy group, not just its identity component, even if M is not simply connected. The point here is that this is a special property of dimensions $\neq$ multiple of 4. If we say that the *restricted* holonomy group of a manifold is $SU(2)$ or $SU(4)$, this *does not mean* that it is a complex manifold. But in dimension 10, if the *restricted* holonomy is $SU(5)$, then the manifold *has to be a Kähler manifold*. The paper [14] discusses this issue in detail.

Such a manifold, whether it is simply connected or not, is a complex manifold (see [26, 27]), and by [26, Theorem 1] and the main result of [27], the space of harmonic spinors (which coincides with the space of parallel spinors in our case since the manifolds are Ricci-flat), has complex dimension 2. Recall (see [10, Chapter III, §10] that in dimension 2 the index of the Dirac operator D is nothing but $\dim_{\mathbf{C}}(\mathrm{Ker} D)$ (mod 2) since $\mathcal{C}\ell_1 \cong \mathbf{C}$. Bott periodicity gives the same fact for all dimensions $8k+2$. This would complete the proof in the case when the original manifold M is simply connected, so that $M = \widetilde{M}$. However, this case is very special due to [26, Theorem 1]. Indeed, the dimension of the space of harmonic spinors is equal to 2 even after factorization by a finite group.

In particular, it means that for any finite group π acting holomorphically on a Kähler manifold $\widetilde{M}$ (the universal cover of M), the kernel $\mathrm{Ker} D$ is always a trivial $\mathbf{R}\pi$-module. Thus $\alpha(M) = 0$. So we have a contradiction in Case (a).

Case (b). If $\alpha(M) \neq 0$ then, again, the holonomy group is special. But a Ricci-flat compact locally irreducible *Spin* manifold of special holonomy with $n =$ a multiple of 4 **must** have a cyclic fundamental group by the theorem of B. McInnes [12]. Again, we have a contradiction. □

Theorem 2.4 *Let M be a compact locally irreducible rigidly scalar-flat Spin manifold with $\dim M \geq 5$. Then M belongs to* **exactly** *one of the following classes.*

- **(I)** *$\alpha(M) = 0$, and yet M has a finite fundamental group and admits no metric of positive scalar curvature. That is, M violates the Gromov-Lawson-Rosenberg Conjecture.*
- **(II)** *M has $\dim M = 4k$, and $\pi_1(M)$ is finite cyclic.*

Proof. In view of Lemma 2.1, we only need to prove that the two classes are disjoint. However, it is known [3] that the Gromov-Lawson-Rosenberg Conjecture is valid when $\pi_1(M)$ is finite cyclic. Hence no rigidly scalar-flat *Spin* manifold can belong to both classes. □

Corollary 2.5 *The Weak Besse Conjecture for Spin manifolds is valid if $\dim M = 4k > 4$ and $\pi_1(M)$ is finite cyclic.*

Corollary 2.6 *Let π be a finite non-cyclic group such that it can be shown that for Spin manifolds of fundamental group π, there exists a metric of positive scalar curvature if and only if $\alpha(M) = 0$. Let M be a compact locally irreducible Spin manifold of dimension ≥ 5 with $\pi_1(M) = \pi$. Then M cannot be rigidly scalar flat.*

Remark 2.7 The Gromov-Lawson-Rosenberg Conjecture is known to be true when a manifold has fundamental group belonging to a short list of groups, see [20] for details. For instance, this is the case when fundamental group is a space form group, i.e. cyclic, quaternionic or generalized quaternionic, see [3].

Corollary 2.8 *Let M be any compact locally irreducible Spin manifold with $\dim M \geq 5$, $\dim M \neq 4k$ and $\alpha(M) \neq 0$. Then the scalar curvature is negative at some point in M.*

Remark 2.9 In other words, these manifolds belong to the class **(N)**. There are many examples of manifolds like this, for instance, Hitchin's exotic spheres [7], or Bérard-Bergery's nine-dimensional example with $\pi_1(M) = \mathbf{Z}_2$; see [21].

Theorem 2.4 means that compact locally irreducible rigidly scalar-flat *Spin* manifolds fall into two disjoint classes: the exotic Class (I), and the Class (II). About the Class (I) we can say little, except that we do not believe it is nonempty. About Class (II), by contrast, we can say a great deal. We introduce some notation. Let n be a multiple of 4. Then we define the following finite set of odd numbers:

$$\mathcal{R}(n) = \begin{cases} \{\text{odd divisors of } 2k+2\} & \text{if } n = 8k+4 \\ \{\text{divisors of } 2k+1\} & \text{if } n = 8k. \end{cases}$$

We will denote by $\mathbf{C}_{\mathrm{vol}}$ a compact simply connected Kählerian complex manifold with a complex volume form ω, and no other non-zero holomorphic form (apart from multiples of ω), and by $\mathbf{C}_{\mathrm{sym}}$ a compact simply connected Kählerian complex manifold with a unique (up to constant scalar multiples) complex symplectic form.

Theorem 2.10 *The following classes of manifolds are identical.*

(1) *Compact locally irreducible rigidly scalar-flat Spin manifolds M with $\dim M = n \geq 5$ equal to a multiple of 4 and with $\pi_1(M)$ cyclic.*

(2) *Compact n-dimensional manifolds, where $n \geq 5$ is a multiple of 4, with linear holonomy from the following list:*

- **(i)** $SU(\frac{n}{2})$.
- **(ii)** $SU(\frac{n}{2}) \rtimes \mathbf{Z}_2$, *if n is a multiple of 8, where the generator of $\mathbf{Z}_2$ acts by complex conjugation.*
- **(iii)** $\mathbf{Z}_r \times Sp(\frac{n}{4})$ *with $r \in \mathcal{R}(n)$.*
- **(iv)** $Spin(7)$, *if $n = 8$.*

(3) *Compact n-dimensional manifolds with $n \geq 5$ a multiple of 4, with structure*

- **(i)** $\mathbf{C}_{\mathrm{vol}}$.
- **(ii)** $\mathbf{C}_{\mathrm{vol}}/\mathbf{Z}_2$, $n = 8k$, *$\mathbf{Z}_2$ is generated by an antiholomorphic map.*
- **(iii)** $\mathbf{C}_{\mathrm{sym}}/\mathbf{Z}_r$ *with $r \in \mathcal{R}(n)$, $\mathbf{Z}_r$ is generated by a holomorphic map.*
- **(iv)** *M is an 8-dimensional simply connected manifold with a closed admissible 4-form (see [8]) and such that its Betti numbers satisfy $b^3 + b^4_+ = b^2 + b^4_- + 25$.*

3 Proofs and Remarks

Proof of Theorem 2.10. (1) $\Longrightarrow$ (3). By Theorem 2.4, $\alpha(M) \neq 0$, so there is a parallel spinor, which means that the holonomy is special, which implies that M is equipped with an Einstein metric, which, finally, must be Ricci flat. By [12, Theorem 2] M is in one of the following classes:

(i) M is a Calabi-Yau manifold, that is, has holonomy $SU(\frac{n}{2})$. When n is a multiple of 4, such a manifold must be simply connected (see [23, p.113]), complex and Ricci-flat. By Bochner theory, any holomorphic form is parallel, hence $SU(\frac{n}{2})$-invariant. By the representation theory of $SU(\frac{n}{2})$, this means that there can be no non-zero holomorphic form other than a complex volume form and its constant multiples. As M is simply connected and Ricci flat, the canonical bundle is trivial, so a complex volume form does indeed exist. Thus M has structure $\mathbf{C}_{\mathrm{vol}}$.

(ii) M is a Calabi-Yau manifold factored by $\mathbf{Z}_2$, generated by an antiholomorphic map. Hence it has structure $\mathbf{C}_{\mathrm{vol}}/\mathbf{Z}_2$. This case occurs only if n is a multiple of 8.

(iii) M is $M^{\mathrm{HK}}/\mathbf{Z}_r$, $r \in \mathcal{R}(n)$, where M^{HK} is hyperKähler. It is well known that compact hyperKähler manifolds are simply connected complex manifolds (see [1]). As the holonomy group of M^{HK} is $Sp(\frac{n}{4})$, and as any holomorphic form must be $Sp(\frac{n}{4})$-invariant, the complex symplectic form must be unique up to constant multiples. Finally,

(iv) M could be 8-dimensional manifold with holonomy $Spin(7)$. Joyce [8] shows that the Betti numbers satisfy the given relation.

(3) $\Longrightarrow$ (2).

(i) Let M be a compact simply connected complex manifold with no non-zero holomorphic form other than a complex volume form ω. Since M is Kählerian and ω trivializes the canonical bundle, Yau's theorem [2] gives us a Ricci-flat Kähler metric on M. As ω is the only non-zero holomorphic form, M is irreducible, and so its holonomy is $SU(\frac{n}{2})$.

(ii) Let $M = \mathbf{C}_{\mathrm{vol}}/\mathbf{Z}_2$. As above, $\mathbf{C}_{\mathrm{vol}}$ has holonomy $SU(\frac{n}{2})$, but since the generator of $\mathbf{Z}_2$ does not act holomorphically, M is not a Kähler manifold, and so its holonomy group is not $SU(\frac{n}{2})$. As $\mathbf{C}_{\mathrm{vol}}$ is simply connected, $\pi_1(M) = \mathbf{Z}_2$ so the holonomy group has precisely 2 components. A straightforward exercise in Lie theory shows that the only two-component subgroup of $SO(n)$ which has $SU(\frac{n}{2})$ as identity component (and which is not contained in $U(\frac{n}{2})$) is $SU(\frac{n}{2}) \rtimes \mathbf{Z}_2$, where the generator of $\mathbf{Z}_2$ acts by complex conjugation.

(iii) If the manifold is Kählerian, and if σ is the complex symplectic form, $\sigma^{\frac{n}{4}}$ trivializes the canonical bundle, so by Yau's theorem $\mathbf{C}_{\text{sym}}$ admits a Ricci-flat Kähler metric. Since σ is unique up to multiples, the holonomy group is $Sp(\frac{n}{4})$. By an averaging argument combined with the Calabi uniqueness theorem (see [15, 16] for details) one can assume that the metric projects to $\mathbf{C}_{\text{sym}}/\mathbf{Z}_r$, which becomes a Kähler manifold with $Sp(\frac{n}{4})$ as the restricted holonomy group. Thus the holonomy group is $(\mathbf{Z}_q \times Sp(\frac{n}{4}))/\mathbf{Z}_2$ for some q. Since a $\mathbf{C}_{\text{sym}}$ manifold is simply connected, $\pi_1(\mathbf{C}_{\text{sym}}/\mathbf{Z}_r) = \mathbf{Z}_r$, so there is a homomorphism from $\mathbf{Z}_r$ onto $\mathbf{Z}_{q/2}$ (we can assume that q is even without loss of generality). Thus $q/2$ divides r, which is odd, so $q/2$ is odd; hence the holonomy group is actually $\mathbf{Z}_{q/2} \times Sp(\frac{n}{4})$. Now suppose $q/2 < r$, and let $k = r/(\frac{q}{2})$. Then $\mathbf{C}_{\text{sym}}/\mathbf{Z}_k$ is a $\frac{q}{2}$-fold covering of $\mathbf{C}_{\text{sym}}/\mathbf{Z}_r$, and so its full holonomy group is precisely $Sp(\frac{n}{4})$. Now let $f : \mathbf{C}_{\text{sym}} \longrightarrow \mathbf{C}_{\text{sym}}$ generate $\mathbf{Z}_r$, so $f^{\frac{q}{2}}$ is a non-trivial fixed-point-free holomorphic map generating $\mathbf{Z}_k$. Since $\text{Hol}(\mathbf{C}_{\text{sym}}/\mathbf{Z}_k) = Sp(\frac{n}{4})$, $f^{\frac{q}{2}}$ preserves σ and all of its powers. But by the representation theory of $Sp(\frac{n}{4})$ (see [1]) σ and its powers are the *only* non-zero holomorphic forms on $\mathbf{C}_{\text{sym}}$. Since all these forms are of *even* degree, the holomorphic Lefschetz number of $f^{\frac{q}{2}}$ is non-zero, which is impossible. Thus actually $\frac{q}{2} = r$ and we have that $\text{Hol}(\mathbf{C}_{\text{sym}}/\mathbf{Z}_r) = \mathbf{Z}_r \times Sp(\frac{n}{4})$.

(iv) Let M be as described. Then Joyce [8] shows that M has a metric of holonomy $Spin(7)$.

(2) $\Longrightarrow$ (1).

(i) Let M be a compact n-dimensional manifold of holonomy $SU(\frac{n}{2})$, where $n \neq 4$ is a multiple of 4. Then M is scalar-flat, simply connected, and $Spin$. It is well-known that such manifolds are rigid (since $\hat{A}(M) = 2 \neq 0$).

(ii) Manifolds M of holonomy $SU(\frac{n}{2}) \rtimes \mathbf{Z}_2$ are double-covered by manifolds of holonomy $SU(\frac{n}{2})$, so it is evident from (i) that they must be scalar-flat, rigid, and have $\pi_1(M) = \mathbf{Z}_2$, which is indeed cyclic. It remains only to prove that these manifolds are $Spin$. For this, observe that *when n is a multiple of* 4, the preimage of $SU(\frac{n}{2}) \rtimes \mathbf{Z}_2$ in $Spin(n)$ is $\{\pm 1\} \times SU(\frac{n}{2}) \rtimes \mathbf{Z}_2$, so $SU(\frac{n}{2}) \rtimes \mathbf{Z}_2$ is a subgroup of $Spin(n)$. Thus if $H(M)$ is a bundle of orthonormal frames with $SU(\frac{n}{2}) \rtimes \mathbf{Z}_2$ as structural group, we can define

$$S(M) = \frac{H(M) \times SO(n)}{SU(\frac{n}{2}) \rtimes \mathbf{Z}_2}, \quad \text{with} \quad S(M)/\{\pm 1\} = SO(M),$$

the corresponding *full* bundle of orthonormal frames. So M is *Spin*. (Actually, $\hat{A}(M) = 1$. See [18] for details of this argument.)

(iii) Manifolds of holonomy $\mathbf{Z}_r \times Sp(\frac{n}{4})$ are of the form $M = \hat{M}/F$ for some manifold $\hat{M}$ of holonomy $Sp(\frac{n}{4})$. It is well known that $\hat{M}$ is scalar-flat and rigid, hence the same is true of $M = \hat{M}/F$. Clearly F acts freely and holomorphically on the hyperKähler manifold $\hat{M}$, but (as is shown in [17]) this implies that F is cyclic. Finally, the preimage of $\mathbf{Z}_r \times Sp(\frac{n}{4})$ in $Spin(n)$ is $\{\pm 1\} \times \mathbf{Z}_r \times Sp(\frac{n}{4})$, so, as above, these manifolds are *Spin*. (Note that $\hat{A}(M) = (1 + \frac{n}{4})/r$.)

(iv) Compact 8-dimensional manifolds with holonomy $Spin(7)$ are necessarily simply connected, *Spin*, and rigidly scalar-flat (since $\hat{A} = 1$).

This completes the proof. □

Remark 3.1 The *Spin* condition plays an essential role throughout this subject. For example, in four dimensions, there are compact manifolds which are not *Spin* and yet are rigidly scalar-flat (e.g. any Enriques surface). Similarly there are rigidly scalar-flat manifolds with non-cyclic fundamental groups (e.g the Hitchin manifold $K3/(\mathbf{Z}_2 \times \mathbf{Z}_2)$.) Such examples are known in all dimensions equal to a multiple of 4; see [15] for their explicit description.

Remark 3.2 When $n = 8$ the possibilities for holonomy groups are

$$\begin{array}{ccccc} SU(4) & = & Spin(6) & \subset & Spin(7), \\ SU(4) \rtimes \mathbf{Z}_2 & = & Spin(6) \sqcup Spin(6) \cdot e_4e_5e_6e_7 & \subset & Spin(7), \\ Sp(2) & = & Spin(5) & \subset & Spin(7), \\ \mathbf{Z}_3 \times Sp(2) & \subset & Spin(2) \cdot Spin(5) & \subset & Spin(7), \\ Spin(7). & & & & \end{array} \tag{1}$$

At present there are known examples of manifolds with all of these holonomy groups (1) except one: $\mathbf{Z}_3 \times Sp(2) \subset Spin(7)$.

We suspect that the techniques developed by Joyce [8] can be extended to give a solution of the following

Open Problem. Construct a closed eight-dimensional manifold with holonomy group $\mathbf{Z}_3 \times Sp(2)$.

Remark 3.3 In the four-dimensional case, the situation is quite different, but again there are obstructions to the existence of metrics of positive scalar curvature; as is well known, these are associated with Seiberg-Witten theory. The following result (for details, see [25, 11, 22]) is relevant.

Theorem 3.4 (Taubes [25], LeBrun [11]) *Let M be a closed, connected, oriented four-manifold with $b_2^+(M) > 1$. If M admits a symplectic structure (in particular, if M is a Kähler manifold), then M does not accept a positive scalar curvature metric.*

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